Stationary Points

Extrema

Extrema are simply points where the direction of a curve changes. For some function $f(x)$, the point $(c, f(c))$ may be a global or local minimum or maximum, or neither. This point:

- is a **global maximum** if $f(c) > f(x)$ for all values of x in the domain
- is a **local maximum** if $f(c) > f(x)$ for all values of x *near* c
- is a **global minimum** if $f(c) < f(x)$ for all values of x in the domain
- is a **local minimum** if $f(c) < f(x)$ for all values of x *near* c

To find extrema we use **stationary points** - places where the gradient is perfectly 0. If the point $c, f(c)$ is a local extremum (maximum or minimum) it must also be a stationary point. However, it's important to note that just because all local extrema are stationary points, not all stationary points are extrema.

Finding Stationary Points With The First Derivative Test

Given the equation $f(x) = 2x^3 - 3x^2$, we can find the stationary points and determine their nature thus:

1. Differentiate the function.
2. Find the stationary points by finding the values of x for which $f\prime(x) = 0$.
3. Divide up the domain with the boundaries at these stationary points. For each section:
   (a) Determine if the derivative is greater than or less than zero.
   (b) Using this, determine if the equation is rising or falling.

$$f\prime(x) = 2x^3 - 3x^2 \tag{1}$$
$$6x^2 - 6x = 0 \therefore x^2 = x \tag{2}$$
$$\therefore x = 0,\ x = 1$$
$$f\prime(x) < 0 : \tag{3}$$
$$f\prime(-1) = 6 \cdot -1^2 - 6 \cdot -1 = 6 + 6 = 12 \tag{3.1}$$
$$\therefore f(x) \text{ is increasing when } x < 0 \tag{3.2}$$

$$0 < f\prime(x) < 1 : \tag{3}$$
$$f\prime(0.5) = 6 \cdot 0.25 - 6 \cdot 0.5 = -1.5 \tag{3.1}$$
$$\therefore f(x) \text{ is decreasing when } 0 < x < 1 \tag{3.2}$$

$$f\prime(x) > 1 : \tag{3}$$
$$f\prime(2) = 6 \cdot 2^2 - 6 \cdot 2 = 24 - 12 = 12 \tag{3.1}$$
$$\therefore f(x) \text{ is increasing when } x < 0 \tag{3.2}$$

If $f\prime(x)$ is greater than zero, then $f(x)$ will always be increasing. If $f\prime(x)$ is less than zero, then $f(x)$ will always be decreasing. However, it is possible that $f\prime(x)$ will be equal to zero, in which case this test is inconclusive.

More formally, for some function $f(x)$ and some stationary point $(c, f(c))$:

If $f\prime(x) > 0$ when $x < c$ and $f\prime(x) < 0$ when $x > c$ then $x = c$ is a local maximum.
If $f\prime(x) < 0$ when $x < c$ and $f\prime(x) > 0$ when $x > c$ then $x = c$ is a local minimum.
If neither of these is true, the test is inconclusive.

The Second Derivative Test

The second derivative test states:

If $f\prime\prime(x) > 0$, then x is a local minimum.
If $f\prime\prime(x) < 0$, then x is a local maximum.
Otherwise, the test is inconclusive.

Examples

Find and classify the stationary point of $f(x) = x^2 - 6x + 3$.

$$
\begin{aligned}
f\prime(x) &= 2x - 6 & \therefore \text{ s.p. at } 2x = 6 \therefore x = 3 \\
f\prime(2) &= 2 \cdot 2 - 6 = -4 & \therefore f(x) \text{ decreasing} \\
f\prime(4) &= 2 \cdot 4 - 6 = 2 & \therefore f(x) \text{ increasing}
\end{aligned}
$$

$\therefore$ local minimum at $x = 3$.

Find and classify the stationary points of $f(x)2x^3 - 6x$.

$$
\begin{aligned}
f\prime &= 6x^2 - 6 & \therefore \text{ s.p. at } 6x^2 = 6 \therefore x = 1, x = -1 \\
x < -1 &: \\
& f\prime(-2) = 6 \cdot -2^2 - 6 = 18 & \therefore f(-2) \text{ is increasing.} \\
-1 < x < 1 &: \\
& f\prime(0) = 6 \cdot 0^2 - 6 = -6 & \therefore f(0) \text{ is decreasing.} \\
x > 1 &: \\
& f\prime(2) = 6 \cdot 2^2 - 6 = 18 & \therefore f(2) \text{ is increasing.}
\end{aligned}
$$

Related Rates

The chain rule can be used to connect rates of change. Given two quantities x and y which are related by $y = f(x)$, if we know how fast x changes (i.e. with respect to time t) as given by $\frac{dx}{dt}$, we can determine how fast y changes using the chain rule.

Given $y = x^2$, if $\frac{dx}{dt} = 5$, we can use this to find $\frac{dy}{dt}$ when $x = 3$.

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\
&= 2x \cdot 5 \\
&= 10x \\
&= 30
\end{aligned}
$$

Examples

The area of a square is growing at $16cm^3/hr$. At what rate is the length of eafch side changing when the length is $4cm$? Let x be the side length and y be the area.

$$
\begin{aligned}
y &= x^2 \\
y\prime &= 2x \\
\frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\
16 &= 2x \cdot \frac{dx}{dt} \\
&= 2 \cdot 4 \cdot \frac{dx}{dt} \\
\frac{16}{8} &= \frac{dx}{dt} \\
\frac{dx}{dt} &= 2
\end{aligned}
$$

Water drops into a circular puddle at a rate of $1cm^3/sec$. The puddle remains a constant 0.1cm deep. Given that the equation for the volume of a cylinder is $V = \frac{\pi r}{10}$, find the rate of increase of the radius.

$$
\begin{aligned}
V\prime &= \frac{2\pi r}{10} \\
\frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\
1 &= \frac{2\pi r}{10} \cdot \frac{dr}{dt} \\
\frac{10}{2\pi r} &= \frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{10}{2 \cdot \pi \cdot 5} \\
&= \frac{1}{\pi}
\end{aligned}
$$