Integration

Indefinite Integrals (Antidifferentiation)

Integration is the reverse of differentiation. Given a function $f(x)$, the antiderivative $F(x)$ is any function which, when differentiated, returns the original $f(x)$.

The basic operation of integration is **add one to the power, divide by the power**. However, as with differentiation, there are special cases. Some of these are listed to the right.

Since the derivative of x^2 is equal to the derivative of x^2+1 or indeed plus any constant, the indefinite integral of $2x$ must include a variable to represent that constant. There are infinitely many possible solutions to $\int 2x\,dx$, which vary only by the value of C.

$f(x)$	$\int f(x)\,dx$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + C$ for $n \neq 1$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + C$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b) + c$ for $ax+b>0$

Finding the Indefinite Integral

Given the equation $\frac{1}{x}+2x$, find its indefinite integral with respect to x for $x>0$.

1. Separate the equation into terms...
2. And integrate each individually.
3. Simplify the result.

$$\begin{aligned}\int (\frac{1}{x}+2x)\,dx &= \int \frac{1}{x}\,dx + 2\int x\,dx && (1)\\ &= \frac{1}{1}\ln(1x) + 2\frac{x^2}{2} && (2)\\ &= \ln(x) + x^2 && (3)\end{aligned}$$

Examples

$$\begin{aligned}\int (3+8x)\,dx &= \int 3\,dx + \int 8x\,dx\\ &= 3\cdot\frac{x}{1} + 8\cdot\frac{x^2}{2}\\ &= 3x + 4x^2 + C\end{aligned}$$

$$\begin{aligned}\int (\cos(7x) + 2e^{5x})\,dx &= \int \cos(7x)\,dx + \int 2e^{-5x}\,dx\\ &= \frac{1}{7}\sin(7x) + 2\cdot -\frac{1}{5}e^{-5x} + C\\ &= \frac{\sin(7x)}{7} - \frac{2}{5}e^{-5x}\end{aligned}$$

$$\begin{aligned}\int 5 - x^2 + \frac{18}{x^4}\,dx &= \int 5\,dx - \int x^2\,dx + \int 18x^{-4}\,dx\\ &= 5x - \frac{x^3}{3} + 18\cdot\frac{x^{-3}}{-3} + C\\ &= 5x - \frac{x^3}{3} - 6x^{-3} + C\end{aligned}$$

$$\begin{aligned}\int \Big(\frac{6}{3t-5} - 4\sin(2t)\Big)dx &= \int \frac{6}{3t-5}dx - \int 4\sin(2t)dx\\ &= \frac{6}{3}\ln(3t-5) - 4\cdot -\frac{1}{2}\cos(2t) + C\\ &= 2\ln(3t-5) + 2\cos(2t) + C\end{aligned}$$

Area Under Curves

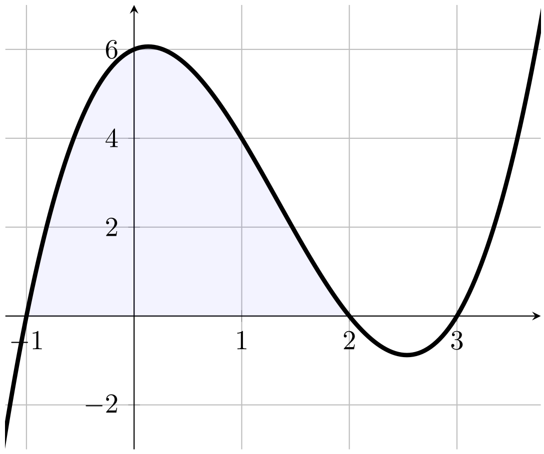
Integration is used to find the area under curves. If $f(x)$ is a continuous function on the interval $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$. In this context, the notation $\int_a^b f(x)dx$ is used to mean "the area under the curve of the function $f(x)$ between $x = a$ and $x = b$."

A **definite integral** is one where the upper and lower bounds are specified.

The graph of the equation

$$y = x^3 - 4x^2 + x + 6 \tag{4}$$

is shown to the right. The area given by $\int_{-1}^2 x^3 - 4x^2 + x + 6dx$ is shaded.



Calculating The Area

Given the definite integral $\int_3^6 x^2dx$, we can evaluate it as follows:

1. Integrate the equation.
2. Rearrange to subtract the integral from itself.
3. Substitute the upper bound into the first instance and the lower bound into the second.
4. Evaluate each.
5. Simplify.

$$\int_3^6 f^x dx = \frac{x^3}{3} \tag{1}$$
$$= \frac{x^3}{3} - \frac{x^3}{3} \tag{2}$$
$$= \frac{6^3}{3} - \frac{3^3}{3} \tag{3}$$
$$= 72 - 9 \tag{4}$$
$$63 \tag{5}$$

Integral Identities

There are a few ways that we can simplify definite integrals.

- for any real numbers $a < b < c$, $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
- $\int_a^a f(z)dx = 0$
- $\int_a^b f(x)dx = \int_b^a f(x)dx$

Examples

Find $\int_{-2}^2 f(x)dx$ for the function $[f(x) = e^x$ for $-2 \leq x \leq 0$ and $f(x) = x^3 + 1$ for $0 \leq x \leq 2]$.

$$\begin{aligned}
\int_{-2}^2 f(x)dx &= \int_{-2}^0 f(x)dx + \int_0^2 f(x)dx \\
&= \int_{-2}^0 e^x dx + \int_0^2 x^3 + 1dx \\
&= [e^x]_{-2}^0 + [\frac{x^4}{4} + x]_0^2 \\
&= e^0 - e^{-2} + (\frac{2^4}{4} + 2) - (\frac{2^4}{4} + 2) \\
&= 1 - e^{-2} + 6 \\
&= 7 - e^{-2}
\end{aligned}$$

Improper Integrals

Improper integrals are definite integrals where one or both of of the bounds are infinite.

The improper integral $\int_a^\infty f(x)dx$ can be written as $\lim_{b\to\infty}\int_a^b f(x)dx$.

These integrals can be calculated by:

1. Find $\int_a^b f(x)dx$ as normal.
2. Evaluate the equation as the infinite term approaches infinity.

$$\int_1^\infty \frac{1}{x^2}dx = \left[\frac{1}{x}\right]_1^\infty \tag{1}$$

$$\lim_{b\to\infty}\frac{1}{1}-\frac{1}{b}=1 \tag{2}$$

Examples

Evaluate $\int_0^\infty 2xdx$, if possible.

$$\begin{aligned}\int_0^\infty 2xdx =& \frac{2x^2}{2}\\ =& [x^2]_0^\infty\\ \lim_{b\to\infty} b^2 =& \infty\\ & \therefore \text{ no integral exists.}\end{aligned}$$