Differentiation

Introduction to Differentiation

Differentiation is a process for finding the gradient of a slope, or the *rate of change* of that slope. The differential of some function $f(x)$ is represented by $f\prime(x)$ or $\frac{d}{dx}f(x)$.

To differentiate an expression, multiply the coefficient by the exponent and subtract 1 from the exponent **- times by the power, take one off the power.** If $f(x) = x^2$, then $f\prime(x) = 2x$.

$$\frac{d}{dx}[f(x) + g(x)] = f\prime(x) + g\prime(x)$$
$$\frac{d}{dx}[f(x) - g(x)] = f\prime(x) - g\prime(x)$$
$$\frac{d}{dx}c \cdot f(x) = c \cdot f\prime(x)$$

Differentiating Special Cases

Certain functions cannot be differentiated by the above rule. Instead, they require special treatment.

$f(x)$	$f\prime(x)$	$f(x)$	$f\prime(x)$
$\sin(ax+b)$	$a\cos(ax+b)$	$\cosh(ax+b)$	$a\sinh(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$	e^{ax}	ae^{ax}
$\sinh(ax+b)$	$a\cosh(ax+b)$	$\ln(ax+b)$	$\frac{a}{ax+b}$

The Product Rule

The Product Rule governs how the product of two derivatives is found. Given $f(x)$ and $g(x)$, the derivative of $f(x) \cdot g(x)$ is:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g\prime(x) + f\prime(x) \cdot g(x)$$

Find the derivative of $h(x) = \frac{5x+1}{3x+2}$.

$$\begin{aligned} h(x) =& x^2 \cdot e^{-4x} \\ f(x) =& x^2 \\ g(x) =& e^{-4x} \\ h\prime(x) =& f(x) \cdot g\prime(x) + f\prime(x) \cdot g(x) \\ =& 2x \cdot e^{-4x} + x^2 \cdot -4e^{-4x} \\ =& 2xe^{-4x} - 4x^{2e^{-4x}} \end{aligned}$$

The Quotient Rule

The Quotient Rule governs how the quotient (or result of division) of two derivatives is found. Given $u = f(x)$ and $v = g(x)$, the derivative of $\frac{f(x)}{g(x)}$ is:

$$\frac{d}{dx}\frac{u}{v} = \frac{v \cdot u\prime - v\prime \cdot u}{v^2}$$

Find the derivative of $h(x) = x^2 \cdot e^{-4x}$.

$$\begin{aligned} u = 5x+1 \quad & u\prime = 5 \\ v = 3x+2 \quad & v\prime = 3 \\ h\prime(x) =& \frac{v \cdot u\prime = v\prime \cdot u}{v^2} \\ =& \frac{5(3x+2) - 3(5x+1)}{(3x+2)^2} \\ =& \frac{7}{(3x+2)^2} \end{aligned}$$

Examples

Differentiate $f(x) = 3\cos(2x+4)$.

$$\begin{aligned} f(x) =& 3\cos(2x+4) \\ f\prime(x) =& 2 \cdot 3 \cdot -\sin(2x+4) \\ =& -6sin(2x+4) \end{aligned}$$

Find the derivative of $y = x^3 \ln(x)$.

$$\begin{aligned} y =& x^3 \ln(x) \\ y\prime =& 3x^2 \cdot \ln(x) + x^3 \cdot \frac{1}{x} \\ =& 3x^2 \ln(x) + x^2 \end{aligned}$$

Differentiate $y = \frac{x}{x^2+4}$

$$\begin{aligned} u = x \quad & u\prime = 1 \\ v = x^2+4 \quad & u\prime = 2x \\ y\prime =& \frac{v \cdot u\prime - v\prime \cdot u}{v^2} \\ =& \frac{x^2+4-2x^2}{(x^2+4)^2} \\ =& \frac{4-x^2}{(x^2+4)^2} \end{aligned}$$

The Chain Rule

A *composition* of functions is the result of nesting them. The functions $f(x) = x^2$, $g(x) = sin(x)$ and $h(x) = e^x$ can all be combined, but depending on the order in which they are combined, the result will be different.

$$\begin{aligned} f(g(x)) =& \sin(x)^2 \\ g(h(x)) =& \sin(e^x) \\ f(f(x)) =& x^4 \\ f(g(h(x))) =& \sin^2(e^x) \end{aligned}$$

Decomposition is the process of separating a single function into multiple simpler ones. This process is illustrated with the decomposition of these functions.

$$y = (x^2 - 3x + 8)^5 \rightarrow y = u^5,\ u = x^2 - 3x + 8$$
$$f(x) = e^{-3x^2} \rightarrow f(u) = e^u,\ u(x) = -3x^2$$
$$y = \cos^2(\ln(x)) \rightarrow y = u^2,\ u = \cos(v),\ v = \ln(x)$$

To differentiate this type of function, we use the Chain Rule.

How To Use The Chain Rule

The Chain Rule states that if a function $f(x)$ can be decomposed to $f(u(x))$ where $f = f(u)$ and $u = u(x)$ then $f\prime(x) = f\prime(u) \cdot u\prime(x)$.

1. Decompose the function into multiple simpler functions.
2. Find the derivative of each of the functions.
3. Substitute in the real values.
4. Simplify the result.

Find the derivative of $y = (x^2 - 3x + 8)^5$.

$$\begin{aligned} y = u^5 \quad & u = x^2 - 3x + 8 && (1) \\ y\prime = 5u^4 \quad & u\prime = 2x - 3 && (2) \\ y\prime =& y\prime(u) \cdot u\prime(x) \\ =& 5u^4 \cdot 2x - 3 && (3) \\ =& 5(x^2 - 3x + 8)^4 \cdot (2x - 3) && (4) \end{aligned}$$

Examples

Find $f\prime(x)$ if $f(x) = (7x - 2)^3$.

$$\begin{aligned} v = u^3 \quad & u = 7x - 2 \\ v\prime = 3u^2 \quad & u\prime = 7 \\ f\prime(x) =& v\prime \cdot u\prime \\ =& 3u^2 \cdot 7 \\ =& 3(7x - 2)^2 \cdot 7 \\ =& 21(7x - 2)^2 \end{aligned}$$

Find $f\prime(x)$ if $f(x) = (5x - 3) - 2$

$$\begin{aligned} v = u^-2 \quad & u = 5x - 3 \\ v\prime = -2u^{-3} \quad & u \\ prime = 5 \\ f\prime(x) =& v\prime \cdot u\prime \\ =& -2u^{-3} \cdot 5 \\ =& -2(5x - 3)^{-3} \cdot 5 \\ =& -10(5x - 3)^{-3} \end{aligned}$$

Find $f\prime(x)$ if $f(x) = e^{-5x^2}$.

$$\begin{aligned} v = e^u \quad & u = -5x^2 \\ v\prime = e^u \quad & u\prime = -10x \\ f\prime(x) =& v\prime \cdot u\prime \\ =& e^u \cdot -10x \\ =& e^{-5x^2} \cdot -10x \\ =& -10xe^{-5x^2} \end{aligned}$$

Implicit Differentiation

It is not always easy or even possible to express y in terms of x. For instance, the equation $y = x^2 - 4x + 6$ makes this trivial, but $y^3 - 5x^2 + 2xy = 0$ proves much trickier. When x and y cannot easily be isolated, this is called an *implicit function*.

To differentiate $y^3 - 5x^2 + 2xy = 0$:

1. Expand the function.
2. Differentiate each group:
   - 2.1. If there is no x, differentiate and multiply by $\frac{dy}{dx}$.
   - 2.2. If there is only x, differentiate it as normal.
   - 2.3. If there are both x and y, use the product rule: $\frac{d}{dx}xy = x\prime y + xy\prime$.
3. Collect and factorise all terms containing $\frac{dy}{dx}$.
4. Transpose to isolate the $\frac{dy}{dx}$ term.
5. Divide by the terms multiplied by $\frac{dy}{dx}$.

Find $\frac{d}{dx}$ of $y^3 - 5x^2 + 2xy = 0$.

$$\frac{d}{dx}y^3 - \frac{d}{dx}5x^2 + \frac{d}{dx}2xy = \frac{d}{dx}0 \qquad (1)$$

$$3y^2\frac{dy}{dx} - 10x + 2x\frac{dy}{dx} + 2y = 0 \qquad (2)$$

$$\frac{d}{dx}y^3 = 2y^2\frac{dy}{dx} \qquad (2.1)$$

$$\frac{d}{dx} - 5x^2 = -10x \qquad (2.2)$$

$$\frac{d}{dx}2xy = 2 \cdot x\prime y + 2 \cdot xy\prime = 2x\frac{dy}{dx} + 2y \qquad (2.3)$$

$$\frac{dy}{dx}(3y^2 + 2x) - 10x + 2y = 0 \qquad (3)$$

$$\frac{dy}{dx}(3y^2 + 2x) = 10y - 2x \qquad (4)$$

$$\frac{dy}{dx} = \frac{10y - 2x}{3y^2 + 2x} \qquad (5)$$

Examples

Given $\ln(y) - \sin(y) + x^2y = 0$, find $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}\ln(y) - \frac{d}{dx}\sin(y) + \frac{d}{dx}x^2y &= 0\\
\frac{1}{y}\frac{dy}{dx} - \cos(y)\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} &= 0\\
\frac{dy}{dx}(\frac{1}{y} - \cos(y) + 2x) + 2xy &= 0\\
\frac{dy}{dx}(\frac{1}{y} - \cos(y) + 2x) &= -2xy\\
\frac{dy}{dx} &= \frac{-2xy}{\frac{1}{y} - \cos(y) + 2x}
\end{aligned}$$

Find $\frac{dy}{dx}$ at $(2, 3)$ of $3x^2 - xy - 2y^2 + 12 = 0$.

$$\begin{aligned}
\frac{d}{dx}3x^2 - \frac{d}{dx}xy - \frac{d}{dx}2y^2 + \frac{d}{dx}12 = \frac{d}{dx}0&\\
x - y - x\frac{dy}{dx} - 4y\frac{dy}{dx} &= 0\\
\frac{dy}{dx}(-x - 4y) - y + 6x &= 0\\
\frac{dy}{dx}(-x - 4y) &= y - 6x\\
\frac{dy}{dx} &= \frac{y - 6x}{-x - 4y}\\
&= \frac{3 - 6 \cdot 2}{-2 - 4 \cdot 3}\\
&= \frac{9}{14}
\end{aligned}$$

Parametric Differentiation

Complex curves like cycloids can use parametric differentiation, a technique which uses one (or more) function for determining x position and one (or more) function for determining y position, connected by the *parameter* t.

The points generated by parametric differentiation can be thought of as the collection of points $(x(t), y(t))$.

The equation of a circle is $x^{2+y^{2=t^2}}$. In parametric form, it could be expressed as $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$ **or** as $x(t) = r\sin(t)$ and $y(t) = r\cos(t)$.

The parametric differential of a set of curves is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{y\prime(t)}{x\prime(t)}$$

The Second Derivative

The second derivative of a parametric curve is given by

$$\frac{d^2y}{sx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{1}{\frac{dx}{dt}}$$

Examples

Find $\frac{dy}{dx}$ of $x(t) = t^2 + 3,\ y(t) = 4t^3$

$$\frac{dy}{dt} = 12t^2 \quad \frac{dx}{dt} = 2t$$
$$\begin{aligned} \frac{dy}{dx} =& \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} \\ =& 12t^2 \cdot \frac{1}{2t} \\ =& \frac{12t^2}{2t} \\ =& 6t \end{aligned}$$

Find $\frac{dy}{dx}$ for $x(t) = t - \sin(t),\ y(t) = 1 - \cos(t)$.

$$\frac{dy}{dt} = \sin(t) \quad \frac{dx}{dt} = 1 - \cos(t)$$
$$\begin{aligned} \frac{dy}{dx} =& \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} \\ =& \sin(t) \cdot \frac{1}{1-\cos(t)} \\ =& \frac{\sin(t)}{1-\cos(t)} \end{aligned}$$

Find $\frac{d^2y}{dx^2}$ for $x(t) = 2t^2,\ y(t) = t^4 - 8t^3$.

$$\frac{dy}{dt} = 4t^3 - 24t^2 \quad \frac{dx}{dt} = 4t$$
$$\begin{aligned} \frac{d^2y}{dx^2} =& \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{1}{\frac{dx}{dt}} \\ \frac{dy}{dx} =& 4t^3 - 24t^2 \cdot \frac{1}{4t} \\ =& \frac{4t^3 - 24t^2}{4t} \\ =& t^2 - 6t \\ \frac{d}{dt} =& 2t - 6 \cdot \frac{1}{4t} \\ =& \frac{2t-6}{4t} \end{aligned}$$