

Differentiation

Introduction to Differentiation

Differentiation is a process for finding the gradient of a slope, or the *rate of change* of that slope. The differential of some function $f(x)$ is represented by $f'(x)$ or $\frac{d}{dx}f(x)$.

To differentiate an expression, multiply the coefficient by the exponent and subtract 1 from the exponent - **times by the power, take one off the power**. If $f(x) = x^2$, then $f'(x) = 2x$.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

$$\frac{d}{dx}c \cdot f(x) = c \cdot f'(x)$$

Differentiating Special Cases

Certain functions cannot be differentiated by the above rule. Instead, they require special treatment.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin(ax + b)$	$a \cos(ax + b)$	$\cosh(ax + b)$	$a \sinh(ax + b)$
$\cos(ax + b)$	$-a \sin(ax + b)$	e^{ax}	ae^{ax}
$\sinh(ax + b)$	$a \cosh(ax + b)$	$\ln(ax + b)$	$\frac{a}{ax+b}$

The Product Rule

The Product Rule governs how the product of two derivatives is found. Given $f(x)$ and $g(x)$, the derivative of $f(x) \cdot g(x)$ is:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Find the derivative of $h(x) = \frac{5x+1}{3x+2}$.

$$h(x) = x^2 \cdot e^{-4x}$$

$$f(x) = x^2$$

$$g(x) = e^{-4x}$$

$$h'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

$$= 2x \cdot e^{-4x} + x^2 \cdot -4e^{-4x}$$

$$= 2xe^{-4x} - 4x^2e^{-4x}$$

The Quotient Rule

The Quotient Rule governs how the quotient (or result of division) of two derivatives is found. Given $u = f(x)$ and $v = g(x)$, the derivative of $\frac{f(x)}{g(x)}$ is:

$$\frac{d}{dx} \frac{u}{v} = \frac{v \cdot u' - v' \cdot u}{v^2}$$

Find the derivative of $h(x) = x^2 \cdot e^{-4x}$.

$$u = 5x + 1 \quad u' = 5$$

$$v = 3x + 2 \quad v' = 3$$

$$\begin{aligned} h'(x) &= \frac{v \cdot u' - v' \cdot u}{v^2} \\ &= \frac{5(3x + 2) - 3(5x + 1)}{(3x + 2)^2} \\ &= \frac{7}{(3x + 2)^2} \end{aligned}$$

Examples

Differentiate $f(x) = 3 \cos(2x + 4)$.

$$f(x) = 3 \cos(2x + 4)$$

$$f'(x) = 2 \cdot 3 \cdot -\sin(2x + 4)$$

$$= -6 \sin(2x + 4)$$

Find the derivative of $y = x^3 \ln(x)$.

$$y = x^3 \ln(x)$$

$$y' = 3x^2 \cdot \ln(x) + x^3 \cdot \frac{1}{x}$$

$$= 3x^2 \ln(x) + x^2$$

Differentiate $y = \frac{x}{x^2 + 4}$

$$u = x \quad u' = 1$$

$$v = x^2 + 4 \quad v' = 2x$$

$$y' = \frac{v \cdot u' - v' \cdot u}{v^2}$$

$$= \frac{x^2 + 4 - 2x^2}{(x^2 + 4)^2}$$

$$= \frac{4 - x^2}{(x^2 + 4)^2}$$

The Chain Rule

A *composition* of functions is the result of nesting them. The functions $f(x) = x^2$, $g(x) = \sin(x)$ and $h(x) = e^x$ can all be combined, but depending on the order in which they are combined, the result will be different.

Decomposition is the process of separating a single function into multiple simpler ones. This process is illustrated with the decomposition of these functions.

To differentiate this type of function, we use the Chain Rule.

$$f(g(x)) = \sin(x)^2$$

$$g(h(x)) = \sin(e^x)$$

$$f(f(x)) = x^4$$

$$f(g(h(x))) = \sin^2(e^x)$$

$$y = (x^2 - 3x + 8)^5 \rightarrow y = u^5, u = x^2 - 3x + 8$$

$$f(x) = e^{-3x^2} \rightarrow f(u) = e^u, u(x) = -3x^2$$

$$y = \cos^2(\ln(x)) \rightarrow y = u^2, u = \cos(v), v = \ln(x)$$

How To Use The Chain Rule

The Chain Rule states that if a function $f(x)$ can be decomposed to $f(u(x))$ where $f = f(u)$ and $u = u(x)$ then $f'(x) = f'(u) \cdot u'(x)$.

1. Decompose the function into multiple simpler functions.
2. Find the derivative of each of the functions.
3. Substitute in the real values.
4. Simplify the result.

Find the derivative of $y = (x^2 - 3x + 8)^5$.

$$y = u^5 \quad u = x^2 - 3x + 8 \quad (1)$$

$$y' = 5u^4 \quad u' = 2x - 3 \quad (2)$$

$$\begin{aligned} y' &= y'(u) \cdot u'(x) \\ &= 5u^4 \cdot 2x - 3 \end{aligned} \quad (3)$$

$$= 5(x^2 - 3x + 8)^4 \cdot (2x - 3) \quad (4)$$

Examples

Find $f'(x)$ if $f(x) = (7x - 2)^3$.

$$\begin{aligned} v &= u^3 \quad u = 7x - 2 \\ v' &= 3u^2 \quad u' = 7 \\ f'(x) &= v' \cdot u' \\ &= 3u^2 \cdot 7 \\ &= 3(7x - 2)^2 \cdot 7 \\ &= 21(7x - 2)^2 \end{aligned}$$

Find $f'(x)$ if $f(x) = (5x - 3)^{-2}$

$$\begin{aligned} v &= u^{-2} \quad u = 5x - 3 \\ v' &= -2u^{-3} \quad u' = 5 \\ f'(x) &= v' \cdot u' \\ &= -2u^{-3} \cdot 5 \\ &= -2(5x - 3)^{-3} \cdot 5 \\ &= -10(5x - 3)^{-3} \end{aligned}$$

Find $f'(x)$ if $f(x) = e^{-5x^2}$.

$$\begin{aligned} v &= e^u \quad u = -5x^2 \\ v' &= e^u \quad u' = -10x \\ f'(x) &= v' \cdot u' \\ &= e^u \cdot -10x \\ &= e^{-5x^2} \cdot -10x \\ &= -10xe^{-5x^2} \end{aligned}$$

Implicit Differentiation

It is not always easy or even possible to express y in terms of x . For instance, the equation $y = x^2 - 4x + 6$ makes this trivial, but $y^3 - 5x^2 + 2xy = 0$ proves much trickier. When x and y cannot easily be isolated, this is called an *implicit function*.

To differentiate $y^3 - 5x^2 + 2xy = 0$:

1. Expand the function.
2. Differentiate each group:
 - (a) If there is no x , differentiate and multiply by $\frac{dy}{dx}$.
 - (b) If there is only x , differentiate it as normal.
 - (c) If there are both x and y , use the product rule: $\frac{d}{dx}xy = x'y + xy'$.
3. Collect and factorise all terms containing $\frac{dy}{dx}$.
4. Transpose to isolate the $\frac{dy}{dx}$ term.
5. Divide by the terms multiplied by $\frac{dy}{dx}$.

Find $\frac{d}{dx}$ of $y^3 - 5x^2 + 2xy = 0$.

$$\frac{d}{dx}y^3 - \frac{d}{dx}5x^2 + \frac{d}{dx}2xy = \frac{d}{dx}0 \quad (1)$$

$$3y^2 \frac{dy}{dx} - 10x + 2x \frac{dy}{dx} + 2y = 0 \quad (2)$$

$$\frac{d}{dx}y^3 = 2y^2 \frac{dy}{dx} \quad (2.1)$$

$$\frac{d}{dx} - 5x^2 = -10x \quad (2.2)$$

$$\frac{d}{dx}2xy = 2 \cdot x'y + 2 \cdot xy' = 2x \frac{dy}{dx} + 2y \quad (2.3)$$

$$\frac{dy}{dx}(3y^2 + 2x) - 10x + 2y = 0 \quad (3)$$

$$\frac{dy}{dx}(3y^2 + 2x) = 10x - 2y \quad (4)$$

$$\frac{dy}{dx} = \frac{10x - 2y}{3y^2 + 2x} \quad (5)$$

Examples

Given $\ln(y) - \sin(y) + x^2y = 0$, find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx} \ln(y) - \frac{d}{dx} \sin(y) + \frac{d}{dx} x^2y &= 0 \\ \frac{1}{y} \frac{dy}{dx} - \cos(y) \frac{dy}{dx} + 2xy + x^2 \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} \left(\frac{1}{y} - \cos(y) + 2x \right) + 2xy &= 0 \\ \frac{dy}{dx} \left(\frac{1}{y} - \cos(y) + 2x \right) &= -2xy \\ \frac{dy}{dx} &= \frac{-2xy}{\frac{1}{y} - \cos(y) + 2x} \end{aligned}$$

Find $\frac{dy}{dx}$ at $(2, 3)$ of $3x^2 - xy - 2y^2 + 12 = 0$.

$$\begin{aligned} \frac{d}{dx}3x^2 - \frac{d}{dx}xy - \frac{d}{dx}2y^2 + \frac{d}{dx}12 &= \frac{d}{dx}0 \\ x - y - x \frac{dy}{dx} - 4y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(-x - 4y) - y + 6x &= 0 \\ \frac{dy}{dx}(-x - 4y) &= y - 6x \\ \frac{dy}{dx} &= \frac{y - 6x}{-x - 4y} \\ &= \frac{3 - 6 \cdot 2}{-2 - 4 \cdot 3} \\ &= \frac{9}{14} \end{aligned}$$

Parametric Differentiation

Complex curves like cycloids can use parametric differentiation, a technique which uses one (or more) function for determining x position and one (or more) function for determining y position, connected by the *parameter* t .

The points generated by parametric differentiation can be thought of as the collection of points $(x(t), y(t))$.

The equation of a circle is $x^2 + y^2 = r^2$. In parametric form, it could be expressed as $x(t) = r \cos(t)$ and $y(t) = r \sin(t)$ or as $x(t) = r \sin(t)$ and $y(t) = r \cos(t)$.

The parametric differential of a set of curves is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

The Second Derivative

The second derivative of a parametric curve is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{1}{\frac{dx}{dt}}$$

Examples

Find $\frac{dy}{dx}$ of $x(t) = t^2 + 3$, $y(t) = 4t^3$

$$\begin{aligned} \frac{dy}{dt} &= 12t^2 \quad \frac{dx}{dt} = 2t \\ \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} \\ &= 12t^2 \cdot \frac{1}{2t} \\ &= \frac{12t^2}{2t} \\ &= 6t \end{aligned}$$

Find $\frac{dy}{dx}$ for $x(t) = t - \sin(t)$, $y(t) = 1 - \cos(t)$.

$$\begin{aligned} \frac{dy}{dt} &= \sin(t) \quad \frac{dx}{dt} = 1 - \cos(t) \\ \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} \\ &= \sin(t) \cdot \frac{1}{1 - \cos(t)} \\ &= \frac{\sin(t)}{1 - \cos(t)} \end{aligned}$$

Find $\frac{d^2y}{dx^2}$ for $x(t) = 2t^2$, $y(t) = t^4 - 8t^3$.

$$\begin{aligned} \frac{dy}{dt} &= 4t^3 - 24t^2 \quad \frac{dx}{dt} = 4t \\ \frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{1}{\frac{dx}{dt}} \\ \frac{dy}{dx} &= 4t^3 - 24t^2 \cdot \frac{1}{4t} \\ &= \frac{4t^3 - 24t^2}{4t} \\ &= t^2 - 6t \\ \frac{d}{dt} &= 2t - 6 \cdot \frac{1}{4t} \\ &= \frac{2t - 6}{4t} \end{aligned}$$