Algebraic And Partial Fractions

Algebraic Fractions

A polynomial is a sum of powers of a variable x, such as $x^3 - 3x^2 - 2x + 1$. An algebraic fraction is the ratio of two polunomials. If the numerator of a fraction is greater than the denominator, it is *improper*, otherwise it is *proper*.

Partial Fractions By Negating Variables

The expression $\frac{2}{x-1} + \frac{5}{x+4}$ can be combined into a single fraction by introducing the common denominator $(x-1)(x+4)$, which results in $\frac{7x+3}{(x-1)(x+4)}$. The method of partial fractions provides a method for reversing that operation.

1. Split the fraction based on the denominator, with A and B as the numerators.
2. Multiply the missing denominator portion.
3. Simplify to a single equation.
4. Equate numerators.
5. Set one set of numerators equal to 0:
6. And solve to get a value for one of the variables.
7. Set the other numerators equal to 0...
8. And solve for the second value.
9. Which gives a solution.

$$\frac{7x+3}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4} \tag{1}$$

$$= \frac{A(x+4)}{(x-1)(x+4)} + \frac{B(x-1)}{(x-1)(x+4)} \tag{2}$$

$$= \frac{A(x+4) + B(x-1)}{(x-1)(x+4)} \tag{3}$$

$$7x + 3 = A(x+4) + B(x-1) \tag{4}$$

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when $x = 1$:

$$7 \cdot 1 + 3 = A(1+4) + B(1-1) \tag{5}$$

$$10 = 5A \therefore A = 2 \tag{6}$$

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when $x = -4$:

$$7 \cdot -4 + 3 = A(-4+4) + B(-4-1) \tag{7}$$

$$-25 = -5B \therefore B = 5 \tag{8}$$

$$\frac{7x+3}{(x-1)(x+4)} = \frac{2}{x-1} + \frac{5}{x+4} \tag{9}$$

Examples

$$\begin{aligned}\frac{5+2}{(x+2)(5x-2)} =& \frac{A}{x+2} + \frac{B}{3x-2}\\ =& \frac{A(3x-2)}{(x+2)(3x-2)} + \frac{B(x+2)}{(x+2)(3x-2)}\\ 5x+2 =& A(3x-2) + B(x+2)\end{aligned}$$

$$\begin{aligned}\text{when } x = -2 :&\\ 5 \cdot -2 + 2 =& A(3 \cdot -2 - 2) + B(-2+2)\\ -8 =& -8A\\ A =& 1\end{aligned}$$

$$\begin{aligned}\text{when } x = \frac{2}{3} :&\\ 5 \cdot \frac{2}{3} + 2 =& A(3 \cdot \frac{2}{3} - 2) + B(\frac{2}{3} + 2)\\ \frac{16}{3} =& \frac{8B}{3}\\ B =& 2\\ \therefore \frac{5x+2}{(x+2)(3x-2)} =& \frac{1}{x+2} + \frac{2}{3x-2}\end{aligned}$$

$$\begin{aligned}\frac{3x+1}{(x-1)(x+1)} =& \frac{A}{x-1} + \frac{B}{x-1}\\ =& \frac{A(x+1)}{(x-1)(x+1)} + \frac{A(x-1}{(x-1)(x+1)}\\ 3x+1 =& A(x+1) + B(x-1)\end{aligned}$$

$$\begin{aligned}\text{when } x = 1 :&\\ 3 \cdot 1 + 1 =& A(1+1) + B(1-1)\\ 4 =& 2A\\ A =& 2\end{aligned}$$

$$\begin{aligned}\text{when } x = -1 :&\\ 3 \cdot -1 + 1 =& A(-1+1) + B(-1-1)\\ -2 =& -2B\\ B =& 1\\ \therefore \frac{3x+1}{(x-1)(x+1)} =& \frac{2}{x-1} + \frac{1}{x+1}\end{aligned}$$

Partial Fractions By Stratifying Exponents

Another method works by equating exponents.

Steps 1-4 are the same as above.

10. Expand both sides of the equation.
11. Factor by powers of x.
12. Separate by powers of x to produce separate equations.
13. Solve as simultaneous equations...
14. ...to find one of the two variables.
15. Take one of the equations from Step 7, and solve for the other variable.
16. Which brings us to the same answer.

$$\begin{aligned} 7x + 3 =& A(x+4) + B(x-1) \\ =& Ax + 4A + Bx - B \qquad (10) \\ =& (A+B)x + (4A - B) \qquad (11) \end{aligned}$$

$$\begin{aligned} [x^1]:\ 7 =& A + B \qquad (12) \\ [x^0]:\ 3 =& 4A - B \\ 7 + 3 =& A + B + 4A - B \qquad (13) \\ 10 =& 5A \\ A =& 2 \qquad (14) \end{aligned}$$

$$\begin{aligned} 7 =& A + B \qquad (15) \\ =& 2 + B \\ B = 5 \\ \therefore \frac{7x+3}{(x-1)(x+4)} =& \frac{2}{x-1} + \frac{5}{x+4} \qquad (16) \end{aligned}$$

Examples

$$\begin{aligned} \frac{10x-7}{(x-4)(2x^2-6x+3)} =& \frac{A}{x+4} + \frac{Bx+C}{2x^{2-6x}} \\ =& \frac{A(2x^2-6x+3)}{(x-4)(2x^2-6x+3)} + \frac{(Bx+C)(x+4)}{(x-4)(2x^2-6x+4)} \\ 10x - 7 =& A(2x^2 - 6x + 3) + (Bx + C)(x+4) \end{aligned}$$

$$\begin{aligned} \text{when } x = 4:\ 10 \cdot 4 - 7 =& A(2 \cdot 4^2 - 6 \cdot 4 + 3) + (B \cdot 4 + C)(4 - 4) \\ 33 =& A(32 - 24 + 3) \\ A =& 3 \end{aligned}$$

$$\begin{aligned} 10x - 7 =& 2Ax^2 - 6Ax + 3A + Bx^2 - 4Bx + Cx - 4C \\ [x^2]:\ 0 =& 2A + B \\ [x^1]:\ 10 =& -6A - 4B + C \\ [x^0]:\ -7 =& 3A - 4C \\ 0 =& 2A + B \\ =& 2 \cdot 3 + B \\ B =& -6 \end{aligned}$$

$$\begin{aligned} 10 =& -6A - 4B + C \\ =& -6 \cdot 3 - 4 \cdot -6 + C \\ =& -18 + 10 - 24 \\ C =& 18 + 10 - 24 \\ =& 4 \end{aligned}$$

$$\frac{10x-7}{(x-4)(2x^2-6x+3)} = \frac{3}{x-4} + \frac{4-6x}{2x^2-6x+3}$$

Combining Methods

It may be necessary to use both techniques to solve more complex fractions. This can be seen when solving the equation $\frac{9x+3}{(x-1)(x^2+x+10)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+10}$.

$$
\begin{aligned}
9x + 3 =& A(x^2 + x + 10) + (Bx + C)(x - 1) && (17)\\
\text{when } x = 1 :& \\
9 \cdot 1 + 3 =& A(1^2 + 1 + 10) + B(\cdot 1 + C)(1 - 1) \\
12 =& 12A \\
A =& 1 && (18)
\end{aligned}
$$

17. Use Steps 1-4 above.
18. Use Steps 5-6 to find A.
19. Use Steps 12-14 to find B.
20. Use Steps 5-6 again to find C.

$$
\begin{aligned}
9x + 3 =& Ax^2 + Ax + 10A + Bx^2 - Bx + Cx - C \\
[x^2] :\ 0 =& A + B \\
[x^1] :\ 9 =& A - B + c \\
[x^0] :\ 3 =& 10A - C \\
B =& - A \\
=& - 1 && (19)
\end{aligned}
$$

$$
\begin{aligned}
\text{when } x = 0 :\ 9 \cdot 0 + 3 =& A(0^2 + 0 + 10) + (B \cdot 0 + C)(0 - 1) \\
3 =& 10 \cdot A + C \cdot -1 \\
3 =& 10 - C \\
C =& 7 && (20)
\end{aligned}
$$

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\therefore \frac{9x + 3}{(x - 1)(x^2 + x + 10} = \frac{1}{z - 1} + \frac{-x + 7}{x^2 + x + 10} \qquad (21)
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